2022-2023 Harvard/University of Global Health Equity
Mathematical Modeling for Infectious Disease Planning

Measles model activity [answers in red**]**

1. **Simple SIR model for measles**

a) Starting with the SIR.bm code provided, we aim to model a measles epidemic in a closed community of 1 million individuals using an SIR framework.
   i. Draw the model diagram below, provide a table of parameter descriptions and write down the corresponding differential equations as shown in the code.

Description of symbols in SIR model for measles

Symbol	**Description**
β	Transmission rate
γ	Recovery rate
S	Number of susceptible individuals
I	Number of infectious individuals
R	Number of recovered individuals
N	Population size

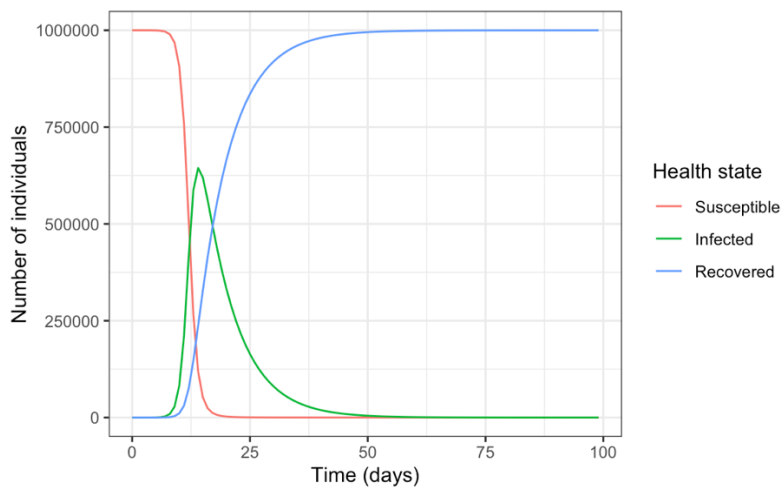
Differential equations for SIR model for measles

$$\frac{dS}{dt} = \frac{-\beta SI}{N},$$
$$\frac{dI}{dt} = \frac{\beta SI}{N} - \gamma I,$$
$$\frac{dR}{dt} = \gamma I,$$
$$N = S + I + R.$$

b) Since measles is quite infectious, in this model we assume that on average, an exposed susceptible person has a .75 probability of getting infected if he or she comes into contact with an infectious person. We also assume that an average person in a population of 1 million has about 12 respiratory contacts a week and that a person with measles remains infectious for about one week. Complete the table of parameter values below.

Symbol	**Description**	**Values**
β	Transmission rate	0.75*12/7
γ	Recovery rate	1/7
N	Population size	1000000

Run the model for a year (365 days) for a daily time step, starting with 1 infectious individual. Look at all three compartments (S, I, R) to see what happens over time. Show the dynamics of the disease over time. **[*Hint*:** *make sure all your parameters are coded using the same time units. Also make sure that your parameters are in the same time units as your time step.*]



c) Now let's consider other infectious diseases of childhood. Consider the following table.

Table: Incubation, latent and infectious periods in days for a variety of viral and bacterial infections:

Infectious Disease	**Mean Incubation period**	**Mean Latent period**	**Mean Infectious period**
Measles	10	7	7
Mumps	19	15	6
Pertussis	8	22	8
Rubella	18	9	11
Diptheria	3	15	3

Chicken pox	15	10	11
Hepatitis B	50	14	20
Poliomyelitis	9	2	2
Influenza	2	2	3
Smallpox	12	9	2
Scarlet fever	2	1	18

If the infectious children were isolated from the other children as soon as they developed symptoms of each of these infectious diseases, how long would the duration of **true infectiousness** for each of these infections be? What percentage of the total infectious period is this true infectiousness duration?
[*Hint: recall definitions of incubation, latent and infectious periods. Also, note that the math of true infectiousness duration = (incubation period – latent period) assumes that symptom onset occurs between start of infectiousness and end of infectiousness. This assumption however does not hold for some diseases like Hep B, poliomyelitis, and smallpox, where symptom onset occurs after the end of the infectious period; that is, people stop being infectious before symptoms show, hence, for those diseases, symptom monitoring and isolation may not help reduce the mean infectious period. We therefore recommend drawing out the timescales for the latent period, infectious period, and incubation period for each disease to help answer this question.*]

Answers for the first disease have been provided.

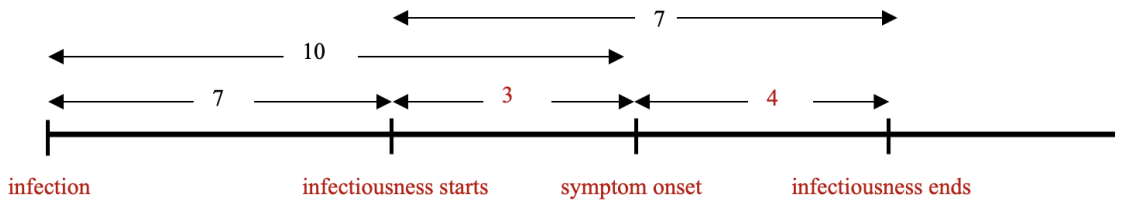
Infectious Disease	**Duration of infectiousness under symptom monitoring and isolation (percent of total infectious period)**
Measles	3 (43% = (3/7) *100%)
Mumps	4 (66%)
Pertussis	0 (0%)
Rubella	9 (82%)
Diptheria	0 (0%)
Chicken pox	5 (45%)
Hepatitis B	20 (100%)
Poliomyelitis	2 (100%)
Influenza	0 (0%)
Smallpox	2 (100%
Scarlet fever	1 (6%)

Recall that:

- *Incubation period: period between infection (exposure) and onset of symptoms*
- *Latent period: period between infection and onset of infectiousness*

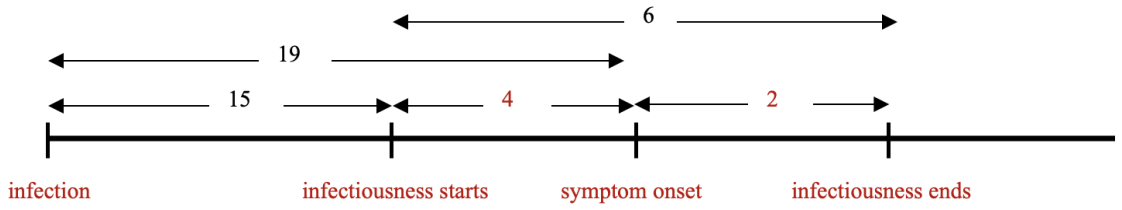
- *Infectious period: period between onset of infectiousness and end of infectiousness*

Measles:

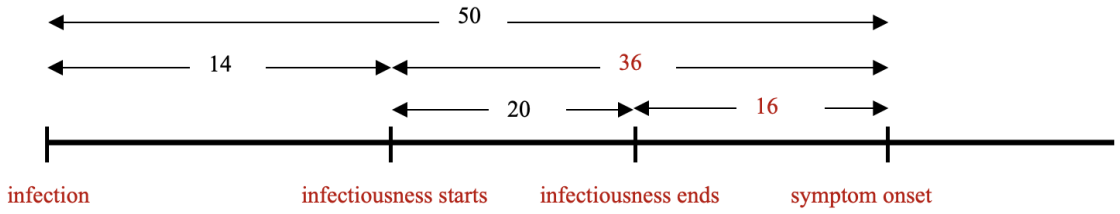


Hence, true infectiousness duration if isolated upon symptom onset = 3 days

Mumps:



Hepatitis B:



Hence, true infectiousness duration if isolated upon symptom onset = 20 days. In this case, isolation upon symptom onset does not change the true infection duration, as symptoms start to occur after the end of infectiousness.

i) For which infectious diseases is isolation most effective?

Symptom monitoring and isolation appears to be the most effective for pertussis, diptheria, influenza, and scarlet fever; i.e., those diseases for which the true duration of infectiousness is less than two days.

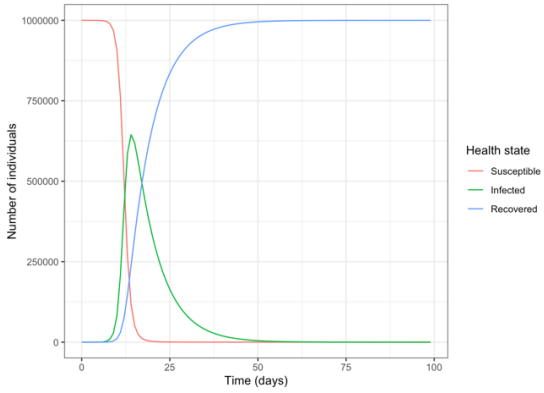
d) Alter the measles model to reflect the practice of isolating kids as soon as the incubation period ends (i.e., as soon as they show symptoms). Which parameter will you modify? How much does that affect the epidemic trajectory? Provide, side by side, the graphs showing the dynamics for the model with isolation and the model without isolation.

We will modify the recovery rate parameter (gamma) to take on the value 1/3, where the denominator is the number of days one spends infectious after accounting for isolation. Recall that rate = 1/duration.

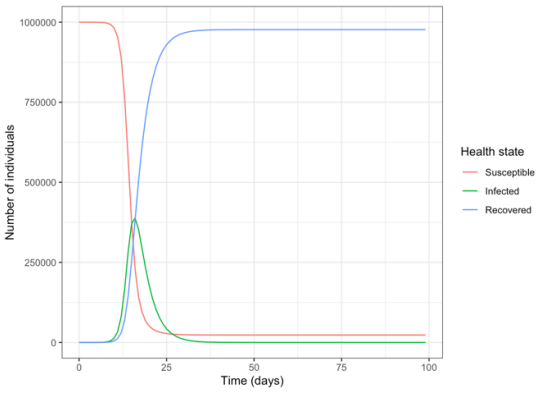
The epidemic growth rate is slower and more susceptibles remain after the epidemic dies out.

Plots showing the first 100 days for average duration of infectiousness = 7 days vs. 3 days:

Measles model with 7-day infectious period



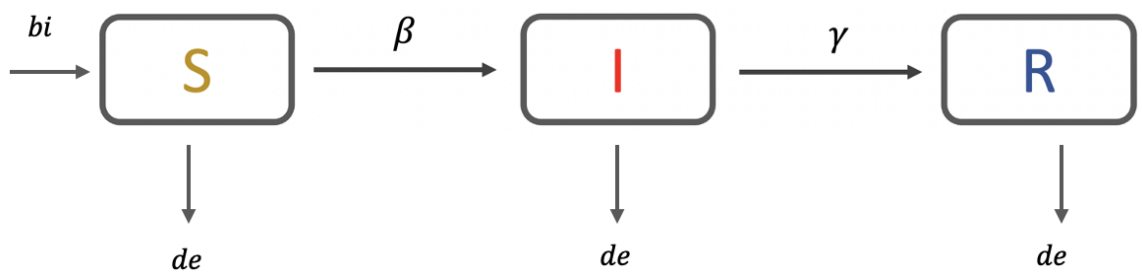
Measles model with 3-day infectious period



2. In a steady-state model, the number of births equals the number of deaths per unit time and population is constant. Return to the SIR model described in Q1b) for measles. Add births (parameter *bi* for birth rate) and deaths (parameter *de* for birth rate) to the model assuming that they are balanced so that we can study the behavior of a simple epidemic in a steady-state situation rather than in a closed population. Use a birth rate and death rate of 0.013 of the population/day.

   Draw the compartmental diagram for the updated model accounting for births and

deaths and write down the differential equations for this model.



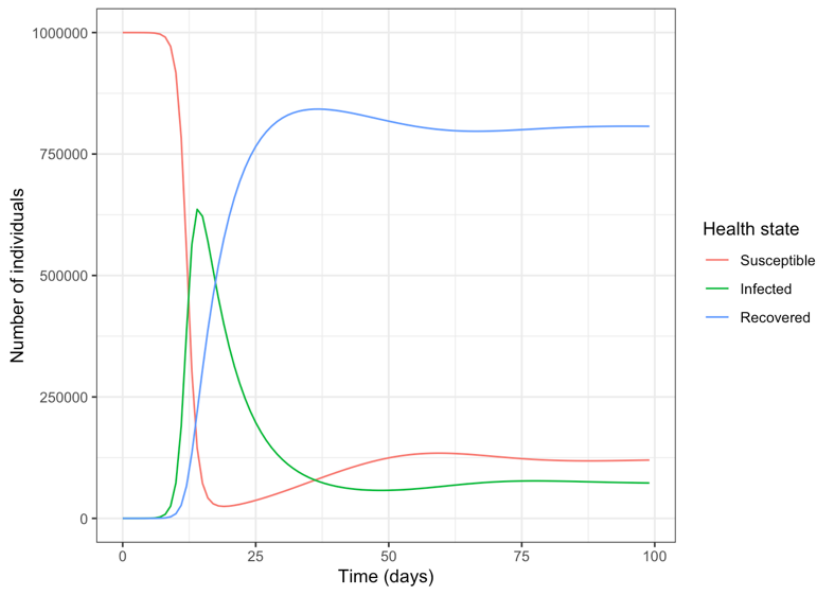
Differential equations for SIR model for measles with births and deaths

$$\frac{dS}{dt} = \frac{-\beta SI}{N} + bi * N - de * S,$$
$$\frac{dI}{dt} = \frac{\beta SI}{N} - \gamma I - de * I,$$
$$\frac{dR}{dt} = \gamma I - de * R,$$
$$N = S + I + R.$$

a) Describe briefly what appears to be happening in terms of timing of the epidemics and the end-behavior in the open model. What is the prevalence of measles at equilibrium?

After initial oscillatory behavior, transmission reaches equilibrium with about ~7.3% prevalence.

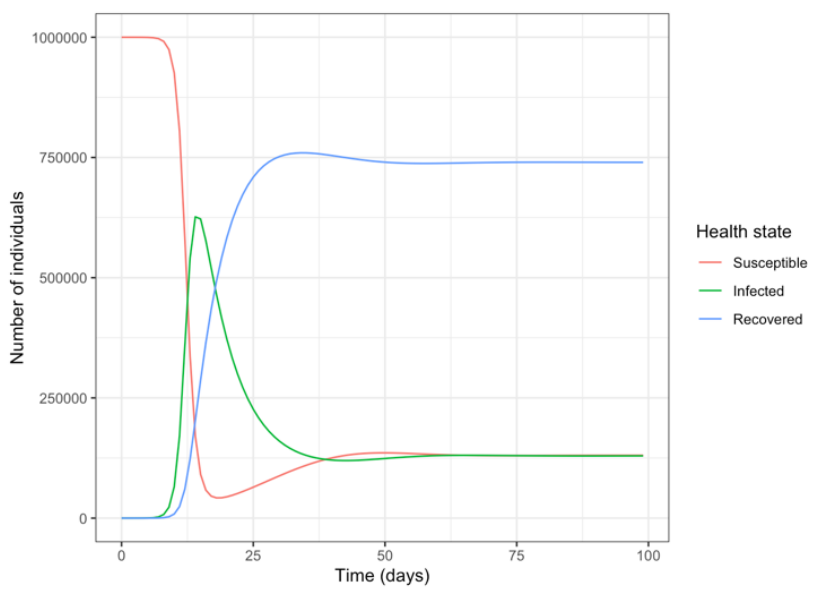
Measles model with births and deaths



b) Does fertility affect epidemic behavior?
In developing countries, both birth rates and mortality are higher than in wealthier countries. Increase the birth and death rate (keeping them equal) from .013 to .025 and observe the change in epidemic trajectory. Briefly describe, graphically and in words, the differences you see in the behavior of measles when fertility and mortality are increased graphically and in words.

Equilibrium prevalence is higher with higher fertility and mortality (~12.9%).

Measles model with increased fertility and mortality



BM Codes

```
{QUESTION 1a)}
{Since measles is quite infectious, in this model we assume that on average, an exposed
susceptible person has a .75 probability of getting infected if he or she comes into contact
with an infectious person. We also assume that an average person in a population of 1
million has about 12 respiratory contacts a week and that a person with measles remains
infectious for about one week. Start the model with 1 infectious individual, and run the
model for a year (365 days) for a daily time step. Look at all three compartments (S, I, R)
to see what happens over time. }

{method for solving equations}
METHOD RK4

{time}
STARTTIME = 0
STOPTIME = 365
DT = 1

{parameter values}
b=0.75*(12/7)
g=1/7
N = 1000000

{differential equations}
d/dt (S) = -b*S*(I/N)
d/dt (I) = b*S*(I/N)- g*I
d/dt(R)=g*I

{initial values}
{Note: start with 1 infected individual, N-1 susceptible individuals and 0 recovered
individuals}
init S = N - 1
init I = 1
init R = 0
```

```
{QUESTION 1d)}
{Alter the measles model to reflect the practice of isolating kids as soon as the incubation
period ends (i.e., as soon as they show symptoms). Which parameter will you modify?
How much does that affect the epidemic trajectory?}
```

```
{method for solving equations}
METHOD RK4

{time}
STARTTIME = 0
STOPTIME = 100
DT = 1

{parameter values, modify value of g}
b=0.75*(12/7)
g=1/3
N = 1000000

{differential equations}
d/dt (S) = -b*S*(I/N)
d/dt (I) = b*S*(I/N)- g*I
d/dt(R)=g*I

{initial values}
init S = N - 1
init I = 1
init R = 0
```

{QUESTION 2a)}
{In a steady-state model, the number of births equals the number of deaths per unit time and population is constant. Return to the SIR model described in Q1 for measles. Add births and deaths to the model assuming that they are balanced so that we can study the behavior of a simple epidemic in a steady-state situation rather than in a closed population. Use a birth rate and death rate of 0.013 of the population/day.

Describe briefly what appears to be happening in terms of timing of the epidemics and the end-behavior in the open model. What is the prevalence of measles at equilibrium?}

```
{method for solving equations}
METHOD RK4

{time}
```

```
STARTTIME = 0
STOPTIME = 100
DT = 1

{parameter values, modify value of g}
b=0.75*(12/7)
g=1/7
N = 1000000
bi = 0.013
de = 0.013

{differential equations}
d/dt (S) = -b*S*(I/N) + bi*N - de*S
d/dt (I) = b*S*(I/N)- g*I - de*I
d/dt(R)=g*I - de*R

{initial values}
init S = N - 1
init I = 1
init R = 0
```

{QUESTION 2b)}
{Does fertility affect epidemic behavior?
In developing countries, both birth rates and mortality are higher than in wealthier countries. Increase the birth and death rate (keeping them equal to one another) from .013 to .025 and observe the change in epidemic trajectory. Briefly describe, graphically and in words, the differences you see in the behavior of measles when fertility and mortality are increased graphically and in words.

}

```
{method for solving equations}
METHOD RK4

{time}
STARTTIME = 0
STOPTIME = 100
DT = 1

{parameter values, modify value of g}
b=0.75*(12/7)
g=1/7
```

```
N = 1000000
bi = 0.025
de = 0.025

{differential equations}
d/dt (S) = -b*S*(I/N) + bi*N - de*S
d/dt (I) = b*S*(I/N)- g*I - de*I
d/dt(R)=g*I - de*R

{initial values}
init S = N - 1
init I = 1
init R = 0
```