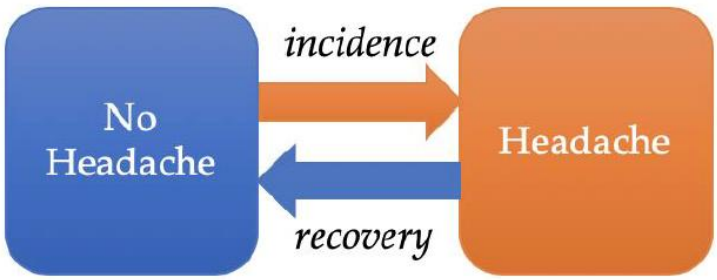
2022-2023 Harvard/University of Global Health Equity
Mathematical Modeling for Infectious Disease Planning[1]

Headaches and bruises: non-infectious compartmental models [answers in red]

1. Draw a model of headaches below to help you think about this exercise, labeling the compartments and rates.



2. Differential equation model of headaches.

a) Write down the differential equations for this model, assuming the population size is N and the population is *closed* (i.e., no births, deaths or migration).
Make sure to define state variables and parameters before writing out your equations.

State variables
NH: no headaches
H: headaches

Parameters
γ = recovery rate
i = incidence

Model equations

$$\frac{dH}{dt} = i * NH - \gamma * H$$

$$\frac{dNH}{dt} = -i * NH + \gamma * H$$

$$N = NH + H$$

b) Open the file *headache.mmd*, which contains Berkeley Madonna (BM) codes of the model equations above.

[1] This exercise was adapted from content for a graduate course at the Harvard T. H. Chan School of Public Health (EPI 501: Dynamics of Infectious Diseases) which was taught by Professors Caroline Buckee and Marc Lipsitch.

c) Read through the code to understand what each part of it does. Vary the two parameters γ and i, to see how they impact the dynamics of headaches. Here, i is the rate of new cases (as a proportion of the NH population) per unit time. γ, the recovery rate, is equal to one over the duration of the disease.

i) Do the following.

a. Increase the original values of γ and i by 10%, 30%, 60% and 90% and run the code for the updated parameter pairs separately. You will be evaluating outcomes for four parameter pairs in total. Comment on the changes in the trend of individuals with headache.

Parameter pair	Increase	Values
(γ, i)	None	(0.5, 0.3)
	10%	(0.55, 0.33)
	30%	(0.65, 0.39)
	60%	(0.80, 0.48)
	90%	(0.95, 0.57)

b. Decrease the original values of γ and i by 10%, 30%, 60% and 90% and run the code for the updated parameter pairs separately. You will be evaluating outcomes for four parameter pairs in total. Comment on the changes in the trend of individuals with headache.

Parameter pair	Decrease	Values
(γ, i)	None	(0.5, 0.3)
	10%	(0.45, 0.27)
	30%	(0.35, 0.21)
	60%	(0.20, 0.12)
	90%	(0.05, 0.03)

ii) What does it mean mathematically when the population is at equilibrium?

Equilibrium is when flows across compartments (health states) are equal in both directions so that the change in the size of the compartment per unit time is 0.

That is, where

$$\frac{dNH}{dt} = \frac{dH}{dt} = i * NH - \gamma * H = 0$$

$$i * NH = \gamma * H$$

iii) Verify that **at equilibrium, the number of individuals with headache** is equal to **duration of headache x incidence**. [Hint: *recovery rate = 1/duration*]

At equilibrium,

$$i * NH = \gamma * H$$

Making H (i.e., the number of individuals with headache) the subject of the equation above, we have

$$H = \underbrace{i * NH}_{\text{incidence}} * \underbrace{\frac{1}{\gamma}}_{\text{duration of disease}}$$

In the equation above, the product of the first two terms is equal to incidence. The expression $1/\gamma$ (i.e., 1/recovery rate) is equal to the duration of headache.